Nonadiabatic noncyclic geometric phase of a spin-$\frac{1}{2}$ particle subject to an arbitrary magnetic field

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We derive a formula of the nonadiabatic noncyclic Pancharatnam phase for a quantum spin-$\frac{1}{2}$ particle subject to an arbitrary magnetic field. The formula is applied to three specific kinds of magneic fields. (i) For an orientated magnetic field, the Pancharatnam phase is derived exactly. (ii) For a rotating magnetic field, the evolution equation is solved analytically. The Aharonov-Anandan phase is obtained exactly and the Pancharatnam phase is computed numerically. (iii) We propose a kind of topological transition in one-dimensional mesoscopic ring subject to an in-plane magnetic field, and then address the nonadiabatic noncyclic effect on this phenomenon.

PACS numbers: 03.65.Bz

I. INTRODUCTION

Berry's phase[1] and its generalization, the Aharonov-Anandan(AA) phase[2], have attracted considerable attention in recent years[3]. It was discovered by Berry that a geometric phase $\gamma_n(C) = i\oint_C \langle n(\vec{R})|\nabla_{\vec{R}}|n(\vec{R})\rangle \cdot d\,\vec{R}$, in addition to the usual dynamic phase, $-\frac{1}{\hbar}\int_0^\tau E_n(\vec{R})dt$, is accumulated on the wavefunction of a quantum system, provided that the Hamiltonian is cyclic and adiabatic. This adiabatic geometric phase has found many applications in physics, particularly in mesoscopic systems where the quantum interference is important. Loss *et al* found that the persistent currents can be induced by the adiabatic Berry phase in a *closed* mesoscopic ring embedded in a static inhomeneous magnetic field[4]. Zhu *et al* proposed a novel experiment to test AA phase in a textured mesoscopic *open* ring subject to a crown-like magnetic field[5]. An interesting kind of topological transition induced by the interference of the adiabatic Berry phase was proposed in Ref.[6]. Moreover, the geometric phase can be generalized to even noncyclic evolution[7–9], and a very recent experiment to test the noncyclic evolution is reported by Wagh *et al*[10].

While dealing with the interference of light, Pancharatnam came up with a brilliant idea regarding a general phase of the evolution for a polarized light[11], which was then generalized to an arbitrary quantum evaluation[7,12]. When a system evolves from an initial state $|\psi(0)\rangle$ to a final state $|\psi(t)\rangle = \hat{U}(t)|\psi(0)\rangle$ with $\hat{U}(t)$ a unitary evolutation operator and $\langle\psi(0)|\hat{U}(t)|\psi(0)\rangle \neq 0$, we refer γ_t as the phase of $|\psi(t)\rangle$ relative to $|\psi(0)\rangle$ once we have

$$\langle\psi(0)|\hat{U}(t)|\psi(0)\rangle = e^{i\gamma_t}|\langle\psi(0)|\hat{U}(t)|\psi(0)\rangle|. \tag{1}$$

For an arbitrary quantum evolution, the geometric Pancharatnam phase can be defined as $\gamma_p = \gamma_t - \gamma_d$, where $\gamma_d = -\frac{1}{\hbar}\int_0^t \langle\psi(t)|\hat{H}|\psi(t)\rangle dt$ is the dynamical phase with $\hat{H}$ as the Hamiltonian of the system.

Consider a quantum system whose normalized state vector $|\psi(t)\rangle$ evolves according to the Schrödinger equation $i\hbar\frac{d}{dt}|\psi(t)\rangle = \hat{H}(t)|\psi(t)\rangle$. Let us define a new state vector $|\phi(t)\rangle$ which differs from $|\psi(t)\rangle$ only in that its dynamical phase factor has been removed. The Pancharatnam phase difference between any two nonorthogonal elements of $\aleph$ can be obtained by the following geodesic rule: If one writes $\langle\phi_1|\phi_2\rangle = \rho exp(i\gamma)$, $\rho > 0$, the phase γ is given by the line integral of A_s along any geodesic lift from $|\phi_1\rangle$ to $|\phi_2\rangle$[7], where $A_s = Im\langle\phi(s)|d/ds|\phi(s)\rangle$ with s as a parameter. Using this rule, we are able to calculate the nonadiabatic noncyclic Pancharatnam phase accumulated in the evolution of a spin-$\frac{1}{2}$ particle subject to an arbitrary magnetic field; It is worth noting that the Pancharatnam phase has physical reality only when the rotated part of the wave function is somehow made to interfere with another part that was not rotated. The formulas to be derived can be used for all two-level systems because any two-level system can be mapped into a system of the spin-$\frac{1}{2}$ in a specific magnetic field[13].

On the other hand, with the advancement of nanotechnology, it is possible to fabricate mesoscopic rings of size within the phase coherence length so that the phase memory is retained by electrons throughout the whole system. In such systems, the electronic quantum transport is significantly affected by the geometric phase which may not be cyclic or adiabatic, However, most theoretical studies of the geometric phase in mesoscopic systems have so far been limited to the cases of adiabatic or cyclic electronic transport. Therefore, it is quite useful and interesting to investigate theoretically the noncyclic nonadiabatic geometric phase and its effect on the electronic transport in mesoscopic systems. Motivated by this, we study the noncyclic nonadiabatic Pancharatnam phase of an electron and discuss the related quantum inference in a mesoscopic ring connected to current leads subject to a magnetic field.

The paper is organized as follows. In Sec. II, we derive

a formula of the noncyclic nonadiabatic geometric Pancharatnam phase for a quantum particle of spin-$\frac{1}{2}$ subject to an arbitrary magnetic field. In Sec. III, the formula is applied to the three systems subject to, respectively, three specific magnetic fields. For an orientated magnetic field, the Pancharatnam phase is derived exactly. For a rotating magnetic field, the evolution equation is solved analytically, and the geometric phase is computed numerically. In particular, a striking topological transition in a mesoscopic ring subject to an in-plane magnetic field is addressed. The paper ends with a brief summary.

II. GENERAL FORMULA

The Hamiltonian for a system of spin-$\frac{1}{2}$ particle subject to an arbitrary magnetic field $\mathbf{B}(t)$ is given by

$$\hat{H}(t) = -\frac{\mu}{2}\mathbf{B}(t)\cdot \vec{\sigma}, \tag{2}$$

where μ is the Bohr magneton, and $\vec{\sigma} = (\sigma_x,\ \sigma_y,\ \sigma_z)$ with $\sigma_{x,y,z}$ as Pauli matrices. The space of states of this system is the projective space $CP^{(1)}$, which is diffeomorphic to the unit sphere S^2 ($CP^{(1)} \simeq S^3/U(1) \simeq S^2$). The point in S^2 associated with an arbitrary state $|\psi\rangle$ of the system is $\mathbf{n} = \langle\psi|\ \vec{\sigma}\ |\psi\rangle$. Reciprocally, for a given vector $bfn \in S^2$, parameterized in a North chart by

$$\mathbf{n} = (n_1, n_2, n_3) = (sin\theta cos\varphi, sin\theta sin\varphi, cos\theta),$$

we can associate this vector with the spin state

$$|\psi\rangle = \begin{pmatrix} e^{-i\varphi/2}cos(\theta/2) \\ e^{i\varphi/2}sin(\theta/2) \end{pmatrix}_\sigma,$$

where subscript σ denotes the spin space. The Schrödinger equation for the state $|\psi(t)\rangle$, $\frac{d}{dt}|\psi(t)\rangle = -\frac{i}{\hbar}\hat{H}(t)|\psi(t)\rangle$, can be expressed in the following form for the vector $\mathbf{n}(t)$: $\frac{d\mathbf{n}(t)}{dt} = -\frac{\mu}{\hbar}\mathbf{B}(t)\times\mathbf{n}(t)$[14]. This equation can be rewritten in a matrix form as

$$\frac{d\mathbf{n}^T(t)}{dt} = \hat{B}_M(t)\mathbf{n}^T(t), \tag{3}$$

with

$$\hat{B}_M(t) = \frac{1}{\hbar}\begin{pmatrix} 0 & \mu B_3(t) & -\mu B_2(t) \\ -\mu B_3(t) & 0 & \mu B_1(t) \\ \mu B_2(t) & -\mu B_1(t) & 0 \end{pmatrix}$$

for $\mathbf{B}(t) = (\ B_1(t),\ B_2(t),\ B_3(t)\ )$, where T represents the transposition of matrix.

The evolution from an initial state $\mathbf{n}(0)$ to a final state $\mathbf{n}(t)$ corresponds to a curve on the sphere S^2. This field-depend curve may be very complicated. A cyclic evolution of the state is represented by a closed curve on the sphere, that is, $\mathbf{n}(\tau) = \mathbf{n}(0)$ with τ as a period of a cycle. Whether the evolution is cyclic or not is dependent on both the magnetic field and the initial state. The evolution of the spin-$\frac{1}{2}$ system is noncyclic in general although it is cyclic in some special cases, which we will discuss later on. The general curve $\mathbf{n}(t)$ can hardly be solved analytically, even though $\mathbf{n}(t)$ may be exactly determined in some special conditions. The solution of Eq.(3) may be written formally as a $\hat{T}$-exponential: $\mathbf{n}^T(t) = \hat{T}exp(\hat{Q}(t))\mathbf{n}^T(0)$ with $\hat{T}$ as the time-ordering operator and $\hat{Q}(t) = \int_0^t \hat{B}_M(t')dt'$. We can ignore the $\hat{T}$-operator if $\hat{B}_M(t)$ at different times commute. Once we find an operator $\hat{S}(t)$ to diagonalize $\hat{Q}(t)$ in the base: $\hat{I}(t) = \hat{S}^{-1}(t)\hat{Q}(t)\hat{S}(t) = diag(\lambda_1(t), \lambda_2(t), \lambda_3(t))$, we have the exact solution:

$$\mathbf{n}^T(t) = \hat{S}(t)e^{\hat{I}(t)}\hat{S}^{-1}(t)\mathbf{n}^T(0). \tag{4}$$

For a general initial state

$$|\phi(t_i)\rangle = \begin{pmatrix} e^{\frac{-i\varphi_i}{2}}cos\frac{\theta_i}{2} \\ e^{\frac{i\varphi_i}{2}}sin\frac{\theta_i}{2} \end{pmatrix}_\sigma,$$

the state at the instant t is

$$|\phi(t)\rangle = \begin{pmatrix} e^{\frac{-i\varphi(t)}{2}}cos\frac{\theta(t)}{2} \\ e^{\frac{i\varphi(t)}{2}}sin\frac{\theta(t)}{2} \end{pmatrix}_\sigma.$$

A unique curve $\mathbf{n}(t)$ on the unit sphere S^2 is determined by the evolution $|\phi(t)\rangle$ with the initial point A of coordinates $\mathbf{n}(t_i) = (sin\theta_i cos\varphi_i, sin\theta_i sin\varphi_i, cos\theta_i)$ and the final point P of coordinates $\mathbf{n}(t_f) = (sin\theta_f cos\varphi_f, sin\theta_f sin\varphi_f, cos\theta_f)$. Then, $|\phi(t)\rangle = \hat{U}(t,0)|\phi(0)\rangle$ with $\hat{U}(t,0) = \hat{T}exp(-\frac{i}{\hbar}\int_0^t \hat{H}(t')dt')$ as the unitary evolution operator which gives a curve $\overbrace{AHP}$ on the unit sphere S^2. If $\langle\phi(0)|\hat{U}(t_f,0)|\phi(0)\rangle$ is not zero, the Pancharatnam phase $\gamma_p(t_f)$ is defined by

$$\langle\phi(0)|\hat{U}(t_f,0)|\phi(0)\rangle = e^{i\gamma_p(t_f)}|\langle\phi(0)|\hat{U}(t_f,0)|\phi(0)\rangle|. \tag{5}$$

Clearly, $\gamma_p(t_f)$ recovers the AA phase γ_{AA} if $\mathbf{n}(t_f) = \mathbf{n}(0)$ for $t_f = \tau > 0$[2]. For a noncyclic evolution, we can introduce a specific unitary operator $\hat{U}_c(\tau, t_f)$ which makes $\mathbf{n}(\tau) = \mathbf{n}(0)$ after the evolution $|\phi(\tau)\rangle = \hat{U}_c(\tau,t_f)|\phi(t_f)\rangle$, and thus we have

$$\begin{aligned}\langle\phi(0)|\hat{U}(t_f,0)|\phi(0)\rangle &= \langle\phi(0)|\hat{U}_c^+(\tau,t_f)\hat{U}_c(\tau,t_f)\hat{U}(t_f,0)|\phi(0)\rangle \\ &= \langle\phi(0)|\hat{U}_c^+(\tau,t_f)|\phi(\tau)\rangle \\ &= \langle\phi(0)|\hat{U}_c^+(\tau,t_f)|\phi(0)\rangle e^{i\gamma_{AA}(\tau)}. \end{aligned} \tag{6}$$

If $\langle\phi(0)|\hat{U}_c^+(\tau,t_f)|\phi(0)\rangle$ is real and positive, it is clear from Eqs.(5) and (6) that the Pancharatnam phase for the noncyclic evolution is given by the AA phase of the specific cyclic evolution C determined by the operator $\hat{U}_c(\tau,t_f)\hat{U}(t_f,0)$, *i.e.*,

$$\gamma_p(t_f) = \gamma_{AA}(\tau). \tag{7}$$

We now consider a special evolution operator $\hat{U}_g(\tau, t_f)$ which makes the state pass from P to A along the shortest path $\overparen{PSA}$ (*i.e.*, the geodesic curve) in the unit sphere of S^2. Then, $\overparen{AHP}$ and $\overparen{PSA}$ forms a closed curve C on the surface S^2. The geometric phase for this cycle is determined from the surface area S_C closed by the curve C[15], *i.e.*,

$$\gamma_{AA}(\tau) = -\frac{1}{2}S_C = -\frac{1}{2}\int_{Surface} \mathbf{n} \cdot d\mathbf{S}.$$

The surface integral can be transformed into a line integral by introducing a vector field $\mathbf{A}_s$ such that $\nabla \times \mathbf{A}_s = -\mathbf{n}/2$ on the surface. It can be found that the vector potential $\mathbf{A}_s(\mathbf{n}) = (\frac{n_2}{2(1+n_3)}, \frac{-n_1}{2(1+n_3)}, 0)$ describing the field of a monopole satisfies the requirement[16]. Therefore, we have

$$\begin{aligned}\gamma_{AA}(\tau) &= \oint_C \mathbf{A}_s \cdot d\mathbf{n} \\ &= -\frac{1}{2}\int_0^{tf} \frac{n_1 \dot{n}_2 - n_2 \dot{n}_1}{1+n_3} dt + \int_{\overparen{PSA}} \mathbf{A}_s \cdot d\mathbf{n}, \quad (8)\end{aligned}$$

where the dot denotes the time derivative, and the second line integral is performed along the shorter geodesic curve from P to A. The equation to describe the geodesic curve through point P to A can be expressed as

$$tg\theta = \frac{-\kappa}{\eta cos\varphi + \zeta sin\varphi}, \quad (9)$$

with

$$\begin{aligned}\eta &= n_2(t_i)n_3(t_f) - n_3(t_i)n_2(t_f), \\ \zeta &= -n_1(t_i)n_3(t_f) + n_3(t_i)n_1(t_f), \\ \kappa &= n_1(t_i)n_2(t_f) - n_2(t_i)n_1(t_f).\end{aligned}$$

Substituting Eq. (9) into $\int_{\overparen{PSA}} \mathbf{A}_s \cdot d\mathbf{n}$, we obtain

$$\int_{\overparen{PSA}} \mathbf{A}_s \cdot d\mathbf{n} = arctg \frac{sin(\varphi_f - \varphi_i)}{ctg\frac{\theta_f}{2} ctg\frac{\theta_i}{2} + cos(\varphi_f - \varphi_i)}. \quad (10)$$

The evolution curve determined from the above operator $\hat{U}_g(\tau, t_f)$ is the geodesic curve, which indeed ensures $\langle\phi(0)|\hat{U}_g^+(\tau, t_f)|\phi(0)\rangle$ to be real and positive[7,9]. Therefore, we have, from Eqs. (7), (8) and (10),

$$\begin{aligned}\gamma_p(t_f) = &-\frac{1}{2}\int_0^{tf} \frac{n_1 \dot{n}_2 - n_2 \dot{n}_1}{1+n_3} dt \\ &+ arctg \frac{sin(\varphi_f - \varphi_i)}{ctg\frac{\theta_f}{2} ctg\frac{\theta_i}{2} + cos(\varphi_f - \varphi_i)}. \quad (11)\end{aligned}$$

Equation (11) is a central result of this paper, which provides a very useful formula for computing the noncyclic nonadiabatic geometric phase for any two-level system. We emphasize that Eq.(11) can be used to any evolution of a spin-$\frac{1}{2}$ particle subject to an arbitrary magnetic field $\mathbf{B}(t)$.

III. APPLICATIONS TO THREE SPECIFIC SYSTEMS

We now apply Eq.(11) to systems subject to an orientated magnetic field, a rotating magnetic field, and a rotating plus a constant magnetic field.

A. An orientated magnetic field

The simplest system is that a spin-$\frac{1}{2}$ particle is subject to an orientated magnetic field, which can be written as $\mathbf{B} = (0, 0, B_3)$. The Pancharatnam phase for this system can be obtained straightforwardly even for time-dependent B_3 because the magnetic matrix $\hat{B}_M(t)$ at different times commute. One can find that

$$\hat{S}(t)e^{\hat{I}(t)}\hat{S}^{-1}(t) = \begin{pmatrix} cos\varphi_t & -sin\varphi_t & 0 \\ sin\varphi_t & cos\varphi_t & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

where $\varphi_t = -\frac{2\mu}{\hbar}\int_0^t B_3(t')dt'$. Thus, for the initial state $\mathbf{n}(0) = (sin\theta_i cos\varphi_i, sin\theta_i sin\varphi_i, cos\theta_i)$, we have $\mathbf{n}(t) = (sin\theta_i cos(\varphi_i + \varphi_t), sin\theta_i sin(\varphi_i + \varphi_t), cos\theta_i)$ from Eq.(4). Therefore, it is straightforward from Eq.(11) to find

$$\gamma_p(t) = -\frac{\varphi_t}{2}(1 - cos\theta_i) + arctg \frac{sin\varphi_t}{ctg^2\frac{\theta_i}{2} + cos\varphi_t}. \quad (12)$$

We can rewrite Eq.(12) as

$$tg[\gamma_p(t) - \frac{\varphi_t}{2}cos\theta_i] = -tg\frac{\varphi_t}{2}cos\theta_i,$$

which recovers the result for the constant magnetic field B_3 reported in Ref.[10]. This noncyclic geometric phase was indeed detected in a well-performed polarized neutron interferometric experiment.

B. A rotating magnetic field

Consider a spin-$\frac{1}{2}$ quantum particle in a rotating magnetic field. The Hamiltonian of the system is Eq.(2) with the magnetic field given by

$$\mathbf{B} = (B_0 cos\omega t, B_0 sin\omega t, B_1), \quad (13)$$

where B_0 and B_1 are constants.

The *adiabatic and cyclic* Berry phase for this system has been found to be $\mp\frac{1}{2}\Omega_C$ with $\Omega_C = 2\pi(1 - cos\alpha)$ as the solid angle that C subtends to the center of the unit sphere[1], where $\alpha = arctg(B_0/B_1)$ is the fixed tilt angle. A general evolution follows a nonadiabatic, and even a noncyclic one. The magnetic matrix $\hat{B}_M$ can now be expressed as

$$\hat{B}_M(t) = \begin{pmatrix} 0 & \omega_1 & -\omega_0 sin\omega t \\ \omega_1 & 0 & \omega_0 cos\omega t \\ \omega_0 sin\omega t & -\omega_0 cos\omega t & 0 \end{pmatrix},$$

with $\omega_i = \mu B_i/\hbar$. Though the matrices $\hat{B}_M(t)$ at different times do not commute, we can still solve this problem exactly. Let us introduce a new vector

$$\mathbf{u}^T(t) = \begin{pmatrix} cos\omega t & sin\omega t & 0 \\ -sin\omega t & cos\omega t & 0 \\ 0 & 0 & 1 \end{pmatrix} \mathbf{n}^T(t). \tag{14}$$

Equation (3) for $\mathbf{n}^T(t)$ can be replaced by an equivalent equation

$$\frac{d}{dt}\mathbf{u}^T(t) = \hat{B_u}\mathbf{u}^T(t),$$

with $\mathbf{u}^T(0) = \mathbf{n}^T(0)$ and

$$\hat{B_u} = \begin{pmatrix} 0 & \omega+\omega_1 & 0 \\ -(\omega+\omega_1) & 0 & \omega_0 \\ 0 & -\omega_0 & 0 \end{pmatrix}.$$

Note that the matrices $\hat{B_u}$ at different times commute because of its time-independence, from Eq.(4) and (14), the curve $\mathbf{n}(t)$ is derived exactly as

$$\mathbf{n}^T(t) = \begin{pmatrix} cos\omega t & -sin\omega t & 0 \\ sin\omega t & cos\omega t & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} sin^2\chi + cos^2\chi cos\omega_s t & cos\chi sin\omega_s t & \frac{1}{2}sin2\chi(1-cos\omega_s t) \\ -cos\chi sin\omega_s t & cos\omega_s t & sin\chi sin\omega_s t \\ \frac{1}{2}sin2\chi(1-cos\omega_s t) & -sin\chi sin\omega_s t & cos^2\chi + sin^2\chi cos\omega_s t \end{pmatrix} \mathbf{n}^T(0), \tag{15}$$

where $\omega_s = \sqrt{\omega_0^2 + (\omega+\omega_1)^2}$ and $\chi = arctg\frac{\omega_0}{\omega_1+\omega}$. From Eq.(15) and Eq.(11), the Pancharatnam phase can be readily computed analytically or numerically, which will be useful in studing the interference effect on the nonadiabatic noncyclic electronic transport across a mesoscopic Aharonov-Bohm ring connected to the current leads[17] (see also, e.g., Sec. IIIA).

For a cyclic evolution, the above result can be further simplified. The evolution can be cyclic if the frequencies ω_s and ω are commensurable, that is, $\omega_s = \frac{m\omega}{k}$ with m and k as irrational integers. Under this condition, the corresponding Pancharatnam phase accumulated in one cycle with the initial state

$$|\phi(0)\rangle = \begin{pmatrix} e^{-i\varphi_i/2}cos(\theta_i/2) \\ e^{i\varphi_i/2}sin(\theta_i/2) \end{pmatrix}_\sigma$$

can be obtained explicitly

$$\gamma_p(\tau) = \gamma_{AA}(\tau) = m\pi(1-cos\beta) - k\pi(1-cos\chi cos\beta), \tag{16}$$

where $\tau = 2k\pi/\omega = 2m\pi/\omega_s$, and $cos\beta = cos\theta_i cos\chi + sin\theta_i sin\chi cos\varphi_i$. If we define an effective magnetic field in the spherical coordinates $\mathbf{B}_{eff}(t) = (B_{eff}, \chi, \omega t)$ with $B_{eff} = \hbar\omega_s/\mu$, β is just a constant angle between the vectors $\mathbf{n}(t)$ and $\mathbf{B}_{eff}(t)$[18]. Note that the evolution given by Eq.(15) is basically the superposition of two rotations. The first one is that the effective magnetic field rotates around the z axis with the angle χ and the angular frequency ω. The second one is the spin precession around the direction of the effective magnetic field with an angle β and a precession angular frequency ω_s. The combination of the two rotations leads to a cyclic evolution only if the frequencies ω_s and ω are commensurable. Obviously, the geometric phases induced by the first and second rotations are respectively the second and first terms of the RHS of Eq.(16). In the adiabatic limit, we have $\chi \to \alpha$ ($\omega/\omega_1 \to 0$), and $\beta \to 0$ ($\varphi_i \to 0$ and $\theta_i \to \chi$) because $\mathbf{n}(t)$ aligns with $\mathbf{B}_{eff}(t)$. Therefore, the adiabatic Berry phase is recovered.

The Pancharatnam phase $\gamma_p(t_f)$ (and $\xi(t_f)$ defined later) versus the time t_f is plotted in Fig.1 with $\varphi_i = \pi/6$, $\theta_i = 5\pi/12$, $\alpha = \pi/3$, $\omega = 50\ Hz$, and $\omega_s = 2\omega$ (solid line), $\sqrt{3}\omega$ (dotted line), respectively. If we define a function $\xi(t_f) = \gamma_p(t_f) - \gamma_p(\eta\tau)$ with $\eta = Int[t_f/\tau]$, we can see from the inset of Fig.1 that $\xi(t_f)$ is a periodic function of t_f with period $\tau = 2\pi/\omega$ for $\omega_s = 2\omega$. Also $\gamma_{AA} = \gamma_p(t_f) - \gamma_p(t_f - \tau) = 0.61$ is the AA phase for the cyclic evolution. However, the dotted line in Fig.1 does not have the above properties because the evolution is not cyclic during finite time.

C. Topological transition in a mesoscopic ring subject to an in-plane magnetic field

Recently, Lyanda-Geller investigated the adiabatic Berry phase induced by the spin-orbit interaction in low dimensional or lowered symmetry conductors, and proposed an interesting phenomenon: topological transition[6]. Here, we propose that this phenomenon may occur in a mesoscopic ring subject to an in-plane magnetic field, which may be easier to be observed. As an application of Eq.(11), we also analyze whether or not this topological transition exists in nonadiabatic noncyclic cases.

Consider a mesoscopic ring with radius r connected to current leads in a static magnetic field, as shown in Fig.2. We assume that the motion of electrons in the whole system is ballistic, however, we include the spin-flip processes induced by the inhomogeneous magnetic field, which is a big merit to consider the Pancharatnam phase rather than the cyclic AA phase or the adiabatic Berry phase, where an artificial restriction that spin-up and spin-down electrons traverse the ring independently is required[4–6].

An incoming electron wave incident from the left lead is splitted into two beams at the left junction and recombined at the right junction into the outgoing wave through the right lead. As a consequence, the motion of spin-$\frac{1}{2}$ electron in the textured ring is equiva-

lent to a quantum spin-$\frac{1}{2}$ in a rotating magnetic field in time. For a beam of electron wave with Fermi velocity $V_f = \hbar k_f/m_e$, where k_f is the Fermi wave vector and m_e is the effective electron mass, the time for electrons to traverse ballistically one round in the ring is $t_0 = \frac{2\pi r}{V_f}$, which is the interval that the electron moves in the magnetic field[5]. In this situation, the Pancharatnam phase mentioned above in addition to the usual Aharonov-Bohm(AB) phase due to the coupling of electrons to the conventional electromagnetic gauge potential, is accumulated on the electron wavefunction. In such a system, the quantum transport is significantly affected by the AB phase and Pancharatnam phase. We assume for simplicity the ring to be symmetric. Following the method originally given by Büttiker, Imry, and Azbel[19], the transmission coefficient affected by the geometric phase can be obtained as

$$T_g = \frac{2\epsilon^2 sin^2(k_f\pi r)(1+cos\gamma)}{[a^2+b^2cos\gamma-(1-\epsilon)cos(2k_f\pi r)]^2+\epsilon^2 sin^2(2k_f\pi r)}, \tag{17}$$

where $a = \pm(\sqrt{1-2\epsilon}-1)/2$, $b = \pm(\sqrt{1-2\epsilon}+1)/2$ with $0 \le \epsilon \le 1/2$, and $\gamma = 2\pi\phi_{AB}/\phi_0 + \gamma_p$ is the total geometric phase with ϕ_{AB} as the AB flux and $\phi_0 = h/e$ as the usual flux quantum. Here the parameter ϵ stands for the coupling strength of the ring to two leads, and $\epsilon = 0$ in the weak coupling limit while $\epsilon = 1/2$ in the strong coupling limit.

The time-dependent Hamiltonian describing the spin motion in Fig.2 is given by[20]

$$\hat{H} = g\mu[\sigma_x(-B_t sin\omega_f t + B_x) + \sigma_y B_t cos\omega_f t], \tag{18}$$

where $\omega_f = 2\pi/t_0$ and g is the gyromagnetic ratio. A natural basis for $\hat{H}$ consists of $|n_+(t)\rangle$ and $|n_-(t)\rangle$ that satisfy $\hat{H}(t)|n_j(t)\rangle = E_j(t)|n_j(t)\rangle$ $(j = +,-)$ is given by $\langle n_j(t)| = \frac{1}{\sqrt{2}}(1, \frac{E_j}{\hbar(\omega_x + i\omega_t exp(i\omega_f t))})$ with corresponding eigenenergies $E_j = j\hbar\sqrt{\omega_t^2+\omega_x^2-2\omega_t\omega_x sin\omega_f t}$ and $\omega_{t,x} = g\mu B_{t,x}/\hbar$. Within the adiabatic approximation, the Berry phase γ_{Berry} accumulated on the wave function is found to be $\gamma_{Berry} = \pi$ for $\omega_x < \omega_t$, $\gamma_{Berry} = \pi/2$ for $\omega_x = \omega_t$, and $\gamma_{Berry} = 0$ for $\omega_x > \omega_t$[6]. It is interesting to note that the adiabatic Berry phase does not continuously vary with the magnetic field. Substituting the Berry phase into Eq.(17) ($\phi_{AB} = 0$), the transmission coefficient T_g versus magnetic field can be obtained as

$$T_g = \begin{cases} 0, & for\ \omega_x < \omega_t \\ \frac{8sin^2(k_f\pi r)}{1+8sin^2(k_f\pi r)}, & for\ \omega_x = \omega_t \\ 1, & for\ \omega_x > \omega_t \end{cases} \tag{19}$$

in the strong coupling limit. Equation (19) gives a mathematical argument for the existence of a topological transition in this system which characters the destructive ($T_g = 0$) to constructive ($T_g = 1$) interference in quantum transport affected by adiabatic Berry phase. According to the Landauer-Büttiker formula[21], the conductance through the system is $G = (e^2/\hbar)T_g$. Therefore, the conductance as a function of either B_t or B_x has steplike character if the other is fixed. This steplike current-magnetic field character, which is stemmed from the topological geometric phase, is referred to as the topological transition.

Does this topological transition still exist in nonadiabatic noncyclic cases? To answer this question, we compute the Pancharatnam phase and substituted it into Eq.(17) without using the adiabatic approximation or cyclic condition. For the case that the initial state is an eigenstate, the transmission coefficient T_g against ω_x/ω_t for $\frac{\omega_t}{\omega_f} = 100,\ 10,\ 1$ are plotted in Fig.3, where $\omega_f = 10^9 Hz$[20], $k_f r = n + 1/2$ with n a non-negative integer. From Fig.3(a), the rather sharp topological transition occurs at $\frac{\omega_x}{\omega_t} = 1$ for $\frac{\omega_t}{\omega_f} = 100$ under which the adiabatic conditions $\omega_t >> \omega_f$ is well satisfied. However, for $\frac{\omega_t}{\omega_f} = 1,\ 10$(Fig.3(b), (c)) the adiabatic conditions are not well satisfied, we can not observe the topological transition. The above result coincides with a geometric point of view. We can roughly decompose the Pancharatnam phase into two parts, the phase induced by the magnetic field trajectory circuit and the spin precession around the magnetic field. In the adiabatic condition, the later one is approximately zero because the spin direction is along the direction of the magnetic field. Then we only need to analyze the first part. It was pointed out that the adiabatic Berry phase for a spin-$\frac{1}{2}$ particle in a magnetic field is a half of the solid angle that the magnetic field trajectory subtends at degeneracy (*i.e.*, at $\vec{B}= 0$ point) [1]. Then $\gamma_{Berry} = 0$ for $\omega_t < \omega_x$ because the magnetic field trajectory circuit does not enclose the degeneracy. On the other hand, $\gamma_{Berry} = \pi$ for $\omega_t > \omega_x$ since the degeneracy is enclosed and the solid angle of the magnetic circuit is $\pm 2\pi$. In the nonadiabatic noncyclic cases, however, the Pancharatnam phase induced by the spin precession is significant, which oscillates quickly around $\frac{\omega_x}{\omega_t} = 1$, with the first part almost unchanged. Therefore, we may conclude that the Pancharatnam phase induced by the spin precession destroys the topological transition.

Finally, we wish to point out whether or not the topological transition exists in nonadiabatic noncyclic motion may be tested by a well designed mesoscopic experiment, in which B_t may be induced by a long straight current-carrying wire pass through normally the center of the ring as shown in Fig.2. For the ballistic motion in a gold ring with $r \sim 1\mu m$ and $V_f \sim 10^5 m/s$, $g \sim 1$, it is required that the corresponding field should be $\sim 1\ Tesla$ for $\omega_t \sim \omega_f$ and $\sim 10^2\ Tesla$ for $\omega_t \sim 100\omega_f$. If the motion in the gold ring is diffusive, ω_f is replaced by $\omega_D = \frac{l}{2\pi r}\omega_f$ (l is the elastic mean free path), the required magnitude field may be less by a factor $\frac{l}{2\pi r}$ (about two orders) than that predicted for the ballistic case. On the other hand, $g \sim 15$ in a $GaAs$ ring, the required magnetic field may

be less than 10 $Tesla$ in the case $\omega_t/\omega_f = 100$ even for ballistic motion. Therefore, the results reported here may be tested in both ballistic and diffusive conditions.

IV. SUMMARY

A useful formula of the noncyclic nonadiabatic geometric phase for a quantum spin-$\frac{1}{2}$ in an arbitrary magnetic field has been formulated exactly, which can be used in any two-level system. The formula has been applied to three specific kinds of magnetic fields. The evolution equations of the spin-$\frac{1}{2}$ particle in an orientated and in a rotating magnetic fields have been solved respectively, and the Pancharatnam phases are computed. We have also found that the nonadiabatic noncyclic phase has a significant impact on the topological transition in a mesoscopic system.

ACKNOWLEDGMENTS

We gratefully acknowledge helpful discussions with Prof. Hua-Zhong Li and Dr. Shi-Dong Liang. This work is supported by a RGC grant of Hong Kong.

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[20] For a beam of electron wave with Fermi wave vector k_f, the rate for electrons to traverse ballistically one round in the ring is $\omega_f = \hbar k_f/m_e r$ which can in turn be regarded as the angular frequency of the otherwise rotating magnetic field. $\omega_f \sim 10^{11}$ for $V_f \sim 10^5 m/s$, $r \sim 1\mu m$. For a typical diffusive condition, ω_f is replaced by $\omega_D = \frac{l}{2\pi r}\omega_f \sim 10^9$ when $\frac{r}{l} \sim 50/\pi$, *see* L. P. Levy *et al*, Phys. Rev. Lett. **64**, 2074 (1990).

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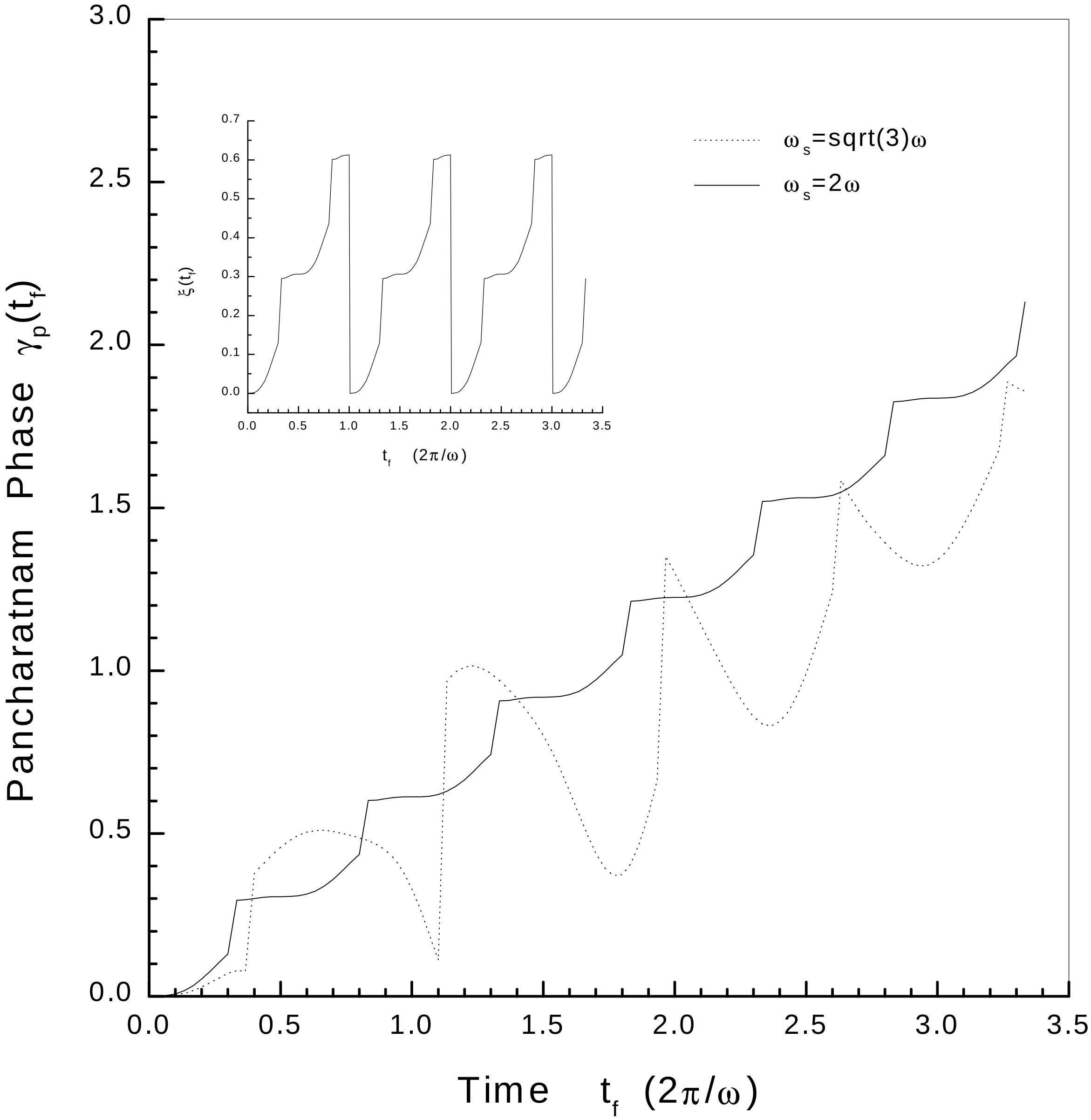
Figure caption

Fig.1 The Pancharatnam phase $\gamma_p(t_f)$ versus the time t_f for $\frac{\omega_s}{\omega} = 2$ (solid line), $\sqrt{3}$ (dotted line). The inset shows that $\xi(t_f)$ versus the time t_f for $\frac{\omega_s}{\omega} = 2$.

Fig.2 A ring connected to current leads in a uniform external magnetic field B_x and a tangent magnetic field B_t, as described by the Hamiltonian (18).

Fig.3 The transmission coefficients T_g versus the ratio ω_x/ω_t for different ω_t/ω_f.



Pancharatnam Phase $\gamma_p(t_f)$
Time t_f $(2\pi/\omega)$
ω_s=sqrt(3)ω
ω_s=2ω
$\xi(t_f)$
t_f $(2\pi/\omega)$

Fig.1

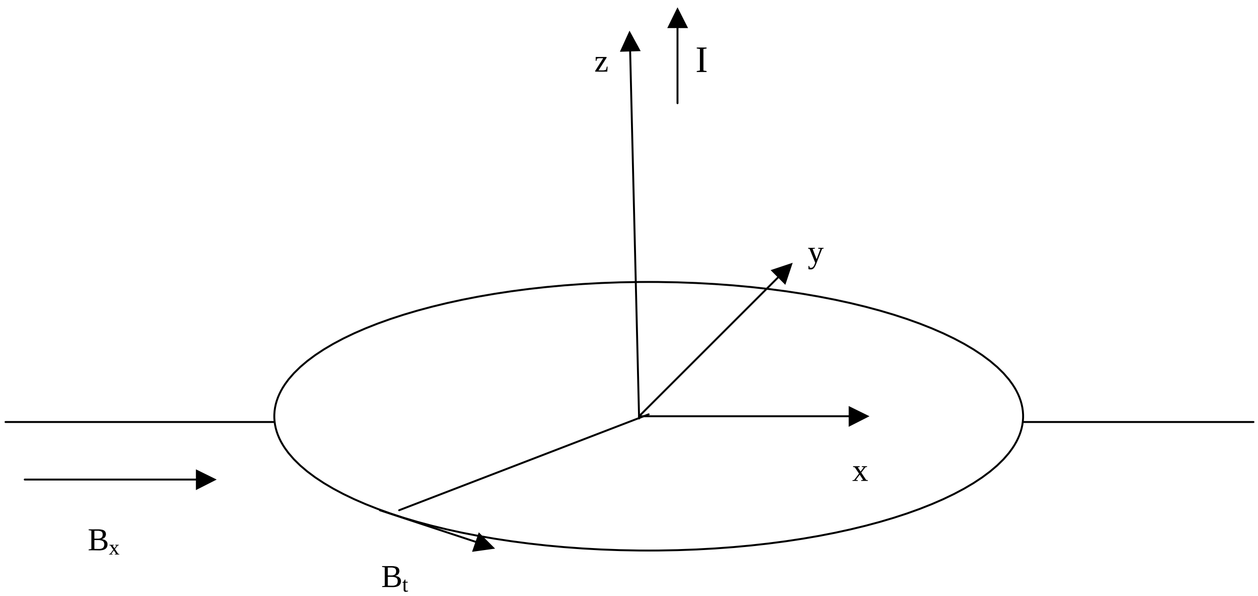
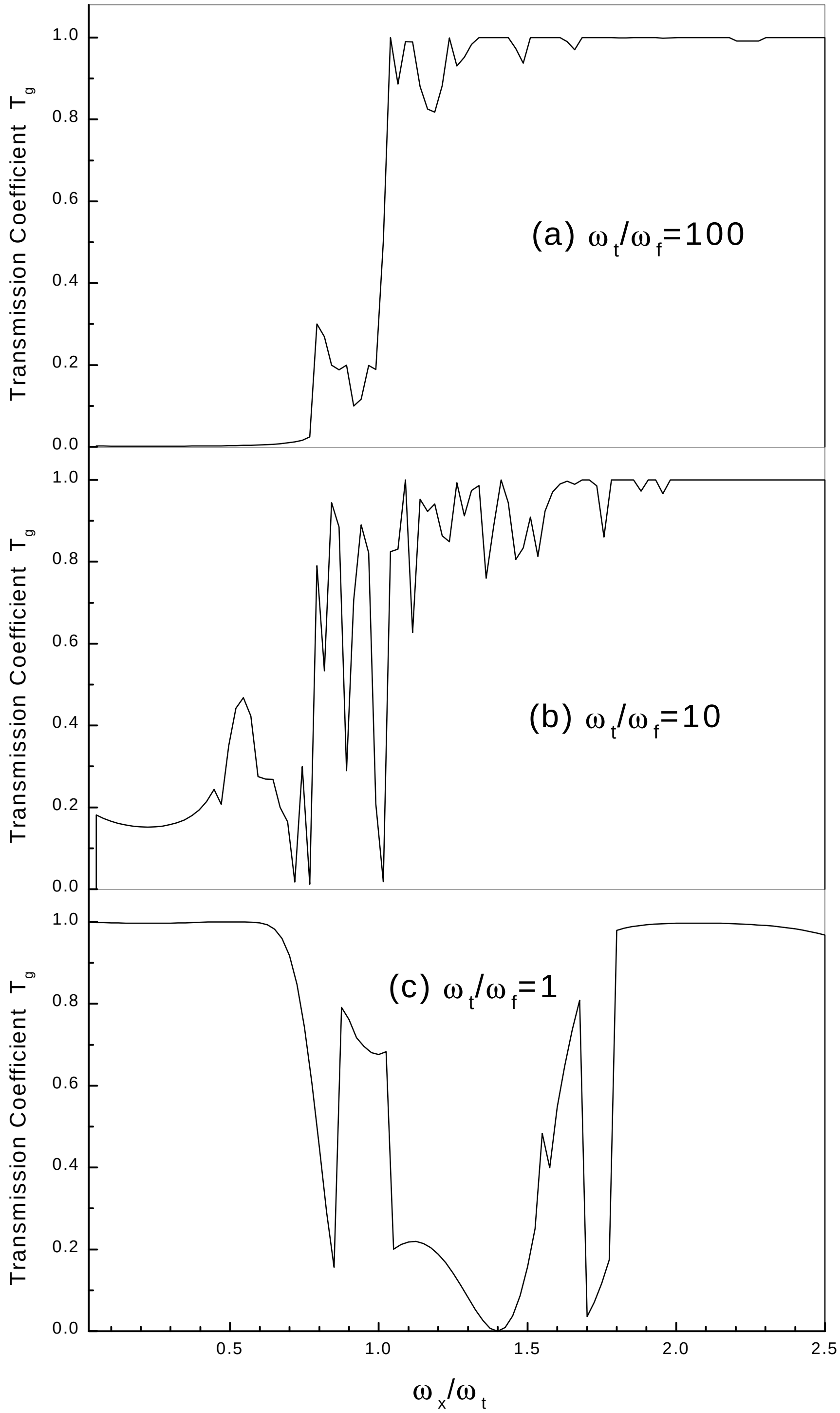


Fig.2



Transmission Coefficient T_g
(a) ω_t/ω_f=100
(b) ω_t/ω_f=10
(c) ω_t/ω_f=1
0.0
0.2
0.4
0.6
0.8
1.0
0.5
1.0
1.5
2.0
2.5
ω_x/ω_t

Fig.3